RESEARCHERID
THOMSON REUTERS

[Kalita, 6(6): June 2019]
DOI- 10.5281/zenodo.3246627

ISSN 2348 – 8034
Impact Factor- 5.070

GLOBAL JOURNAL OF ENGINEERING SCIENCE AND RESEARCHES

GRAM SPECTRUMS OF COXETER ANDREEV'S TETRAHEDRONS AND THEIR EXISTENCE IN SPACES: EUCLIDEAN, SPHERICAL AND HYPERBOLIC

Pranab Kalita
Department of Mathematics, Pandit Deendayal Upadhyaya Adarsha Mahavidyalaya, Darrang-784116, Assam, India

ABSTRACT

In our previous paper, using graph theory and combinatorics, a study on a special type of tetrahedron called Coxeter Andreev's Tetrahedron (CAT) had been facilitated, and it was found that there are exactly 1, 4 and 30 CATs having respectively two edges of order $n \geq 6$, one edge of order $n \geq 6$ and no edge of order $n \geq 6$, $n \in$ upto symmetry. In the present paper, we have studied about the shapes or existence of these $1+4+30=35$ CATs in the spaces: Euclidean, Spherical and Hyperbolic.

Keywords: *Dihedral angles, Coxeter Andreev's Tetrahedron, Gram matrix, Spectrum.*
MSC 2010 Codes: 51F15, 20F55, 51M09.

I. INTRODUCTION

A simplex (in plural, simplexes or simplices) is a generalization [1] of the notion of a triangle or tetrahedron to arbitrary dimensions. A *n*-dimensional polytope P in $X = E^n / S^n / H^n$, with $n > 0$, is a *n*-simplex [9] if and only if P has exactly $n+1$ sides. Specifically, a *n*-simplex is a *n*-dimensional polytope which is the convex hull of its $n+1$ vertices. In particular, a tetrahedron is a 3-dimensional simplex. Tetrahedron is the only 3-simplex convex polyhedron having four faces. The tetrahedron shape has a wide application [2] in engineering and computer science. Tetrahedral mess generation is one of such application. In chemistry, the tetrahedron shape is seen in nature in covalent bonds of molecules. For example, in a methane molecule $\left(CH_4\right)$ or an ammonium ion $\left(NH_4^+\right)$, four hydrogen atoms surround a central carbon or nitrogen atom with tetrahedral symmetry.

Previously, Roland K. W. Roeder's Theorem [25] provides the classification of compact hyperbolic tetrahedron by restricting to non-obtuse dihedral angles. Vinberg proved in [27] that there are no compact hyperbolic coxeter polytopes in H^n when $n \geq 30$. Tumarkin classified the hyperbolic coxeter pyramids in terms of coxeter diagram and John Mcleod generalized it in his article [10]. D. A. Derevnin, at el [26] found the volume of symmetric tetrahedron. Again, in our article [28], using graph theory and combinatorics, a study on a special type of tetrahedron called coxeter Andreev's tetrahedron has been facilitated and it has been found that there are exactly one, four and thirty coxeter Andreev's tetrahedrons having respectively two edges of order $n \geq 6$, one edge of order $n \geq 6$ and no edge of order $n \geq 6$, $n \in$ upto symmetry. Now, in the present paper, we have studied about the shapes or existence of these $1+4+30=35$ CATs in the spaces: Euclidean, Spherical and Hyperbolic.

II. COXETER ANDREEV'S TETRAHEDRONS (CATs)

The following definitions are taken from our previous article [28]

Definition 2.1: The angle between two faces of a polytope, measured from perpendiculars to the edge created by the intersection of the planes is called a *dihedral angle*.

Definition 2.2: A *coxeter* dihedral angle is a dihedral angle of the form $\frac{\pi}{n}$ where, n is a positive integer ≥ 2. A polytope with coxeter dihedral angles is called a *coxeter polytope*.

Definition 2.3: If the dihedral angle of an edge of a polytope is $\frac{\pi}{n}, n$ is a positive number, then n is said to be the order of the edge. We define a trivalent vertex to be of *order* (l,m,n) if the three edges at that vertex are of orders l,m,n.

Definition 2.4: An Andreev's polytope is an abstract polytope which satisfies the following Andreev's conditions [16].

(1) Each dihedral angle α_i is non-obtuse $\left(0<\alpha_i \leq \frac{\pi}{2}\right)$.

(2) Whenever three distinct edges e_i,e_j,e_k meet at a vertex, then $\alpha_i+\alpha_j+\alpha_k>\pi$.

(3) Whenever $\Gamma_p(3)$ intersecting edges e_i,e_j,e_k, then $\alpha_i+\alpha_j+\alpha_k<\pi$.

(4) Whenever $\Gamma_p(4)$ intersecting edges e_i,e_j,e_k,e_l, then $\alpha_i+\alpha_j+\alpha_k+\alpha_l<2\pi$.

(5) Whenever there is a four sided face bounded by edges e_1,e_2,e_3,e_4, enumerated successively, with edges $e_{12},e_{23},e_{34},e_{41}$ entering the four vertices (edge e_{ij} connects to the ends of e_i and e_j), then $\alpha_1+\alpha_3+\alpha_{12}+\alpha_{23}+\alpha_{34}+\alpha_{41}<3\pi$, and $\alpha_2+\alpha_4+\alpha_{12}+\alpha_{23}+\alpha_{34}+\alpha_{41}<3\pi$.

An Andreev's polytope with coxeter dihedral angles is called a *coxeter Andreev's polytope*.

III. GRAM SPECTRUMS OF COXETER ANDREEV'S TETRAHEDRONS AND THEIR EXISTENCE IN SPACES: EUCLIDEAN, SPHERICAL AND HYPERBOLIC

The gram matrix is the most essential and natural tool associated to a simplex. It takes an important role in scientific computing, statistical mechanics and random matrix theory [20]. The geometric properties of a simplex are enclosed in the eigenvalues of a gram matrix. The shape of a simplex is determined by the determinant of its gram matrix. The gram matrix G of a k-simplex in X whose sides are $s_1,s_2,\ldots,s_{k+1}$ is the $(k+1)\times(k+1)$ matrix with ijth entry is $-\cos\theta_{ij}$, θ_{ij} is the angle between the sides s_i and s_j. The gram matrix G is symmetric (real), the eigenvalues of G are real and hence can be ordered, say $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$. The spectrum of a gram matrix is said to be gram spectrum. Let G be a gram matrix with eigenvalues $\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_r$ having respective multiplicities $m_1,m_2,m_3,\ldots,m_r$. Then the gram spectrum of G is written as

$$\sigma(G)=\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_r\\ m_1 & m_2 & m_3 & \ldots & m_r\end{pmatrix} \text{ or } \sigma(G)=\left(\lambda_1^{m_1}, \lambda_2^{m_2}, \lambda_3^{m_3}, \ldots, \lambda_r^{m_r}\right).$$

For a gram matrix G with eigen values $\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n$, the Gram Energy is defined as $GE(G)=\sum_{i=1}^{n}|\lambda_i|$.

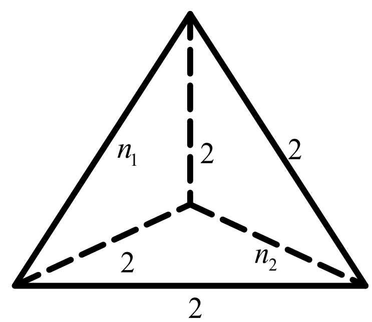
Definition 3.1: A simplex lies in Euclidean, Spherical or Hyperbolic space if the determinant of the gram matrix is 0, positive or negative respectively.

RESEARCHERID
THOMSON REUTERS
[Kalita, 6(6): June 2019] ISSN 2348 – 8034
DOI- 10.5281/zenodo.3246627 Impact Factor- 5.070

Definition 3.2: Consider a compact tetrahedron (Euclidean/Spherical/Hyperbolic) T with dihedral angles A,B,C at the edges adjacent to one vertex, and D,E,F are the dihedral angles opposite to A,B,C respectively. Then the gram matrix of the tetrahedron $T(A,B,C,D,E,F)$ is defined as

$$G=\begin{bmatrix} 1 & -\cos A & -\cos B & -\cos F \\ -\cos A & 1 & -\cos C & -\cos E \\ -\cos B & -\cos C & 1 & -\cos D \\ -\cos F & -\cos E & -\cos D & 1 \end{bmatrix}$$

Theorem 3.3: [28] In a CAT T, if exactly two edges are of order $n \geq 6$, then there exists exactly 1 such T upto symmetry. Refer figure 1.



$$\mathrm{T}_{2n}\text{-}1=[m\geq 6,2,2,n\geq 6,2,2]$$

Figure 1

Result 3.4: Determinant and spectrum of the gram matrix of the CAT $\mathrm{T}_{2n}\text{-}1=[m\geq 6,2,2,n\geq 6,2,2]$ obtained in theorem 3.3 is calculated and listed in table 1. This table also shows the space in which this CAT exists.

Table 1

CAT	Determinant	Spectrum	Space (Ref. Def. 3.1)
$\mathrm{T}_{2n}\text{-}1=[m\geq 6,2,2,n\geq 6,2,2]$	$1-\cos^2\left(\frac{\pi}{n}\right)-\cos^2\left(\frac{\pi}{m}\right)+\cos^2\left(\frac{\pi}{m}\right)\cos^2\left(\frac{\pi}{n}\right)$	$\left(1-\cos\frac{\pi}{m},1+\cos\frac{\pi}{m},1-\cos\frac{\pi}{n},1+\cos\frac{\pi}{n},\right)$	Spherical (Ref. Theorem 3.5)

Theorem 3.5: The CAT $\mathrm{T}_{2n}\text{-}1=[m\geq 6,2,2,n\geq 6,2,2]$ exists in spherical space.

Proof: The determinant of the gram matrix for $\mathrm{T}_{2n}\text{-}1=[m\geq 6,2,2,n\geq 6,2,2]$ is

$$\begin{vmatrix} 1 & -\cos\frac{\pi}{m} & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} \\ -\cos\frac{\pi}{m} & 1 & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} \\ -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} & 1 & -\cos\frac{\pi}{n} \\ -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{n} & 1 \end{vmatrix} = \begin{vmatrix} 1 & -\cos\frac{\pi}{m} & 0 & 0 \\ -\cos\frac{\pi}{m} & 1 & 0 & 0 \\ 0 & 0 & 1 & -\cos\frac{\pi}{n} \\ 0 & 0 & -\cos\frac{\pi}{n} & 1 \end{vmatrix}$$

$$= 1-\cos^2\left(\frac{\pi}{n}\right)-\cos^2\left(\frac{\pi}{m}\right)+\cos^2\left(\frac{\pi}{m}\right)\cos^2\left(\frac{\pi}{n}\right)$$

Let $f(m,n)=1-\cos^2\left(\frac{\pi}{n}\right)-\cos^2\left(\frac{\pi}{m}\right)+\cos^2\left(\frac{\pi}{m}\right)\cos^2\left(\frac{\pi}{n}\right)$. Then $f(m,n)>0, \forall m\geq 6, \forall n\geq 6$ as shown in figure 2. The figure 2 is obtained by keeping m as constant, $m\geq 6$ and $n\geq 6$. By definition 3.1, the CAT $\mathrm{T_{2n}}\text{-}1=[m\geq 6,2,2,n\geq 6,2,2]$ exists in spherical space.

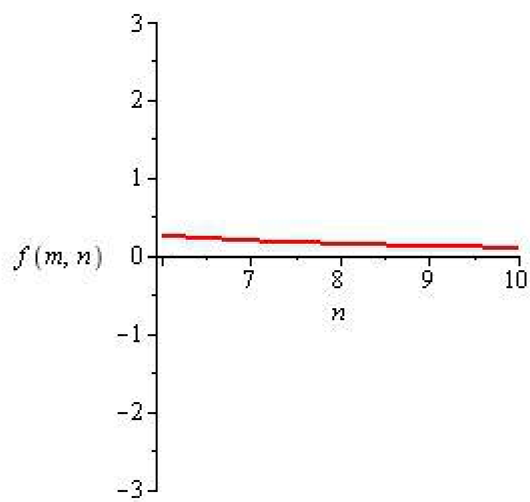
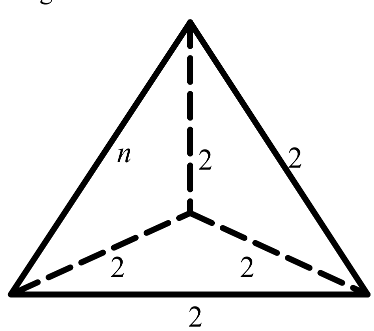
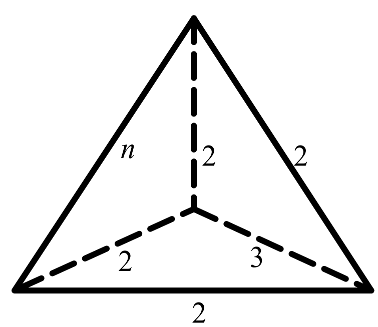


Figure 2

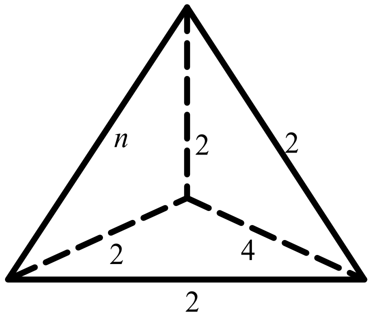
Theorem 3.6: [28] In a CAT T, if exactly one edge is of order $n\geq 6$, then there exists exactly 4 such T upto symmetry. Refer figure 3.



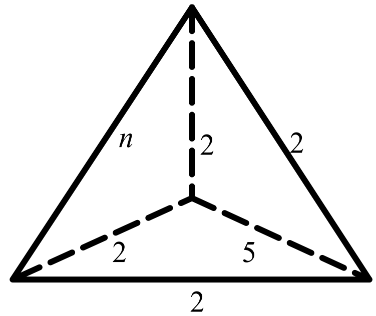
$\mathrm{T_{1n}}\text{-}1=[n\geq 6,2,2,2,2,2]$



$\mathrm{T_{1n}}\text{-}2=[n\geq 6,2,2,3,2,2]$



$\mathrm{T}_{1n}\text{-}3 = [n \geq 6, 2, 2, 4, 2, 2]$



$\mathrm{T}_{1n}\text{-}4 = [n \geq 6, 2, 2, 5, 2, 2]$

Figure 3

Result 3.7: Determinants and spectrums of the gram matrices of the 4 CATs obtained in theorem 3.6 are calculated and listed in table 2. This table also shows the spaces in which these 4 CATs tetrahedrons exist.

Table 2

CATs	Determinant	Spectrum	Space (Ref. Def. 3.1)
$\mathrm{T}_{1n}\text{-}1 = [n \geq 6, 2, 2, 2, 2, 2]$	$1-\cos^2\frac{\pi}{n}$	$\left(1+\sqrt{1-\sin^2\frac{\pi}{n}},\ 1-\sqrt{1-\sin^2\frac{\pi}{n}},\ 1,\ 1\right)$	Spherical (Ref. Theorem 3.8)
$\mathrm{T}_{1n}\text{-}2 = [n \geq 6, 2, 2, 3, 2, 2]$	$\frac{3}{4}-\frac{3}{4}\cos^2\left(\frac{\pi}{n}\right)$	$\left(\frac{1}{2},\ \frac{3}{2},\ 1-\cos\frac{\pi}{n},\ 1+\cos\frac{\pi}{n}\right)$	Spherical (Ref. Theorem 3.9)
$\mathrm{T}_{1n}\text{-}3 = [n \geq 6, 2, 2, 4, 2, 2]$	$\frac{1}{2}-\frac{1}{2}\cos^2\left(\frac{\pi}{n}\right)$	$\left(1+\frac{1}{2}\sqrt{2},\ 1-\frac{1}{2}\sqrt{2},\ 1-\cos\frac{\pi}{n},\ 1+\cos\frac{\pi}{n}\right)$	Spherical (Ref. Theorem 3.10)

ISSN 2348 – 8034
Impact Factor- 5.070

$\mathrm{T}_{1n}\text{-}4=[n\geq 6,2,2,5,2,2]$	$1-\cos^2\left(\frac{\pi}{5}\right)-\cos^2\left(\frac{\pi}{n}\right)+\cos^2\left(\frac{\pi}{n}\right)\cos^2\left(\frac{\pi}{5}\right)$	$\left(1-\cos\frac{\pi}{n}, 1+\cos\frac{\pi}{n}, 1+\cos\frac{\pi}{5}, 1-\cos\frac{\pi}{5}\right)$	Spherical (Ref. Theorem 3.11)

Theorem 3.8: The CAT $\mathrm{T}_{1n}\text{-}1=[n\geq 6,2,2,2,2,2]$ exists in spherical space.

Proof: The determinant of the gram matrix for $\mathrm{T}_{1n}\text{-}1=[n\geq 6,2,2,2,2,2]$ is

$$\begin{vmatrix} 1 & -\cos\frac{\pi}{n} & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} \\ -\cos\frac{\pi}{n} & 1 & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} \\ -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} & 1 & -\cos\frac{\pi}{2} \\ -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} & 1 \end{vmatrix} = \begin{vmatrix} 1 & -\cos\frac{\pi}{n} & 0 & 0 \\ -\cos\frac{\pi}{n} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$=1-\cos^2\left(\frac{\pi}{n}\right)$$

Let $f(n)=1-\cos^2\left(\frac{\pi}{n}\right)$. Then $f(n)>0, \forall n\geq 6$ as shown in figure 4. By definition 3.1, the CAT $\mathrm{T}_{1n}\text{-}1=[n\geq 6,2,2,2,2,2]$ exists in spherical space.

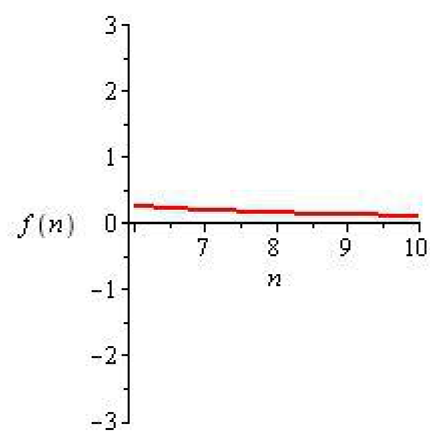


Figure 4

Theorem 3.9: The CAT $\mathrm{T}_{1n}\text{-}2=[n\geq 6,2,2,3,2,2]$ exists in spherical space.

Proof: The determinant of the gram matrix for $\mathrm{T}_{1n}\text{-}2=[n\geq 6,2,2,3,2,2]$ is

 ISSN 2348 – 8034
Impact Factor- 5.070

$$\begin{vmatrix} 1 & -\cos\frac{\pi}{n} & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} \\ -\cos\frac{\pi}{n} & 1 & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} \\ -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} & 1 & -\cos\frac{\pi}{3} \\ -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{3} & 1 \end{vmatrix} = \begin{vmatrix} 1 & -\cos\frac{\pi}{n} & 0 & 0 \\ -\cos\frac{\pi}{n} & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{vmatrix}$$

$$= \frac{3}{4} - \frac{3}{4}\cos^2\left(\frac{\pi}{n}\right)$$

Let $f(n) = \frac{3}{4} - \frac{3}{4}\cos^2\left(\frac{\pi}{n}\right)$. Then $f(n) > 0, \forall n \geq 6$ as shown in figure 5. By definition 3.1, the CAT $T_{1n}\text{-}2 = [n \geq 6, 2, 2, 3, 2, 2]$ exists in spherical space.

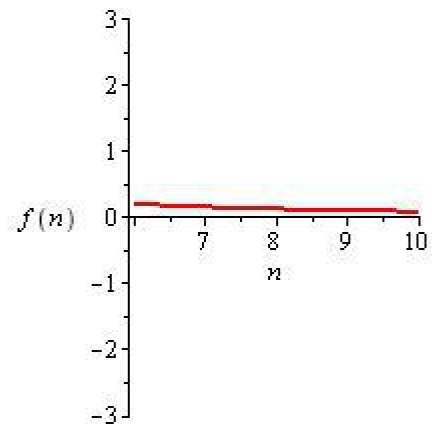


Figure 5

Theorem 3.10: The CAT $T_{1n}\text{-}3 = [n \geq 6, 2, 2, 4, 2, 2]$ exists in spherical space.

Proof: The determinant of the gram matrix for $T_{1n}\text{-}3 = [n \geq 6, 2, 2, 4, 2, 2]$ is

$$\begin{vmatrix} 1 & -\cos\frac{\pi}{n} & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} \\ -\cos\frac{\pi}{n} & 1 & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} \\ -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} & 1 & -\cos\frac{\pi}{4} \\ -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{4} & 1 \end{vmatrix} = \begin{vmatrix} 1 & -\cos\frac{\pi}{n} & 0 & 0 \\ -\cos\frac{\pi}{n} & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2}\sqrt{2} \\ 0 & 0 & -\frac{1}{2}\sqrt{2} & 1 \end{vmatrix}$$

$$= \frac{1}{2} - \frac{1}{2}\cos^2\left(\frac{\pi}{n}\right)$$

Let $f(n) = \frac{1}{2} - \frac{1}{2}\cos^2\left(\frac{\pi}{n}\right)$. Then $f(n) > 0, \forall n \geq 6$ as shown in figure 6. By definition 3.1, the CAT $T_{1n}\text{-}3 = [n \geq 6, 2, 2, 4, 2, 2]$ exists in spherical space.

ISSN 2348 – 8034
Impact Factor- 5.070

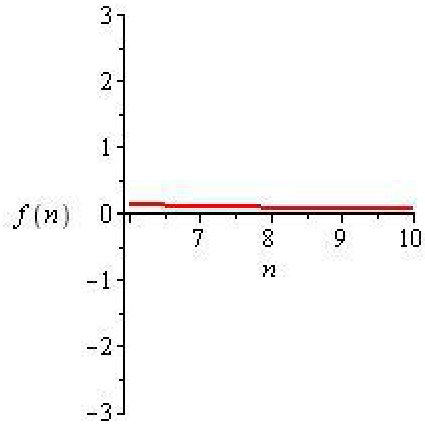


Figure 6

Theorem 3.11: The CAT $T_{1n}\text{-}4=[n\geq 6,2,2,5,2,2]$ exists in spherical space.

Proof: The determinant of the gram matrix for $T_{1n}\text{-}4=[n\geq 6,2,2,5,2,2]$ is

$$\begin{vmatrix} 1 & -\cos\frac{\pi}{n} & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} \\ -\cos\frac{\pi}{n} & 1 & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} \\ -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} & 1 & -\cos\frac{\pi}{5} \\ -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} & -\cos\frac{\pi}{5} & 1 \end{vmatrix} = \begin{vmatrix} 1 & -\cos\frac{\pi}{n} & 0 & 0 \\ -\cos\frac{\pi}{n} & 1 & 0 & 0 \\ 0 & 0 & 1 & -\cos\frac{\pi}{5} \\ 0 & 0 & -\cos\frac{\pi}{5} & 1 \end{vmatrix}$$

$$=1-\cos^2\left(\frac{\pi}{5}\right)-\cos^2\left(\frac{\pi}{n}\right)+\cos^2\left(\frac{\pi}{n}\right)\cos^2\left(\frac{\pi}{5}\right)$$

Let $f(n)=1-\cos^2\left(\frac{\pi}{5}\right)-\cos^2\left(\frac{\pi}{n}\right)+\cos^2\left(\frac{\pi}{n}\right)\cos^2\left(\frac{\pi}{5}\right)$. Then $f(n)>0,\forall n\geq 6$ as shown in figure 7. By definition 3.1, the CAT $T_{1n}\text{-}4=[n\geq 6,2,2,5,2,2]$ exists in spherical space.

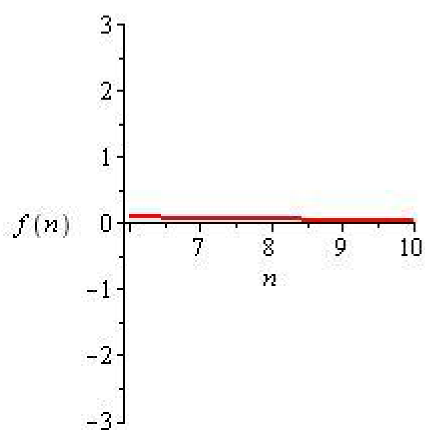


Figure 7

RESEARCHERID
THOMSON REUTERS
[Kalita, 6(6): June 2019] ISSN 2348 – 8034
DOI- 10.5281/zenodo.3246627 Impact Factor- 5.070

Theorem 3.12: [28] In a CAT T, if T has no edge of order $n \geq 6$, then there are exactly 10 such T upto symmetry with at least one vertex is of order $(2,2,2)$. Refer figure 8.

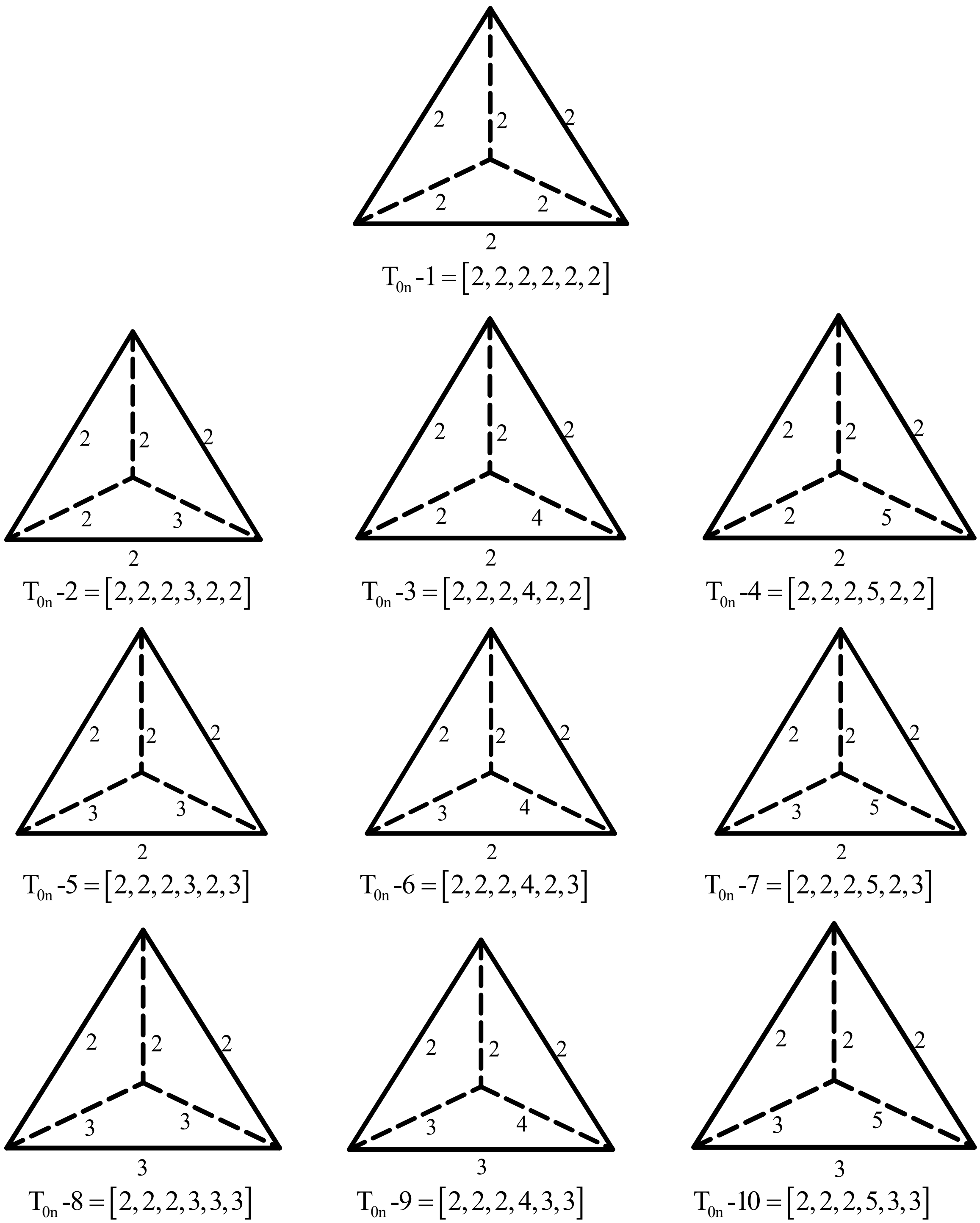


Figure 8

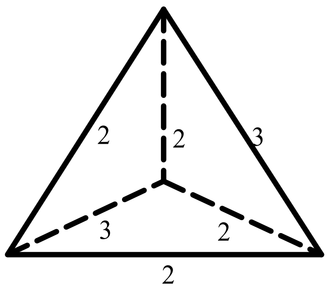
RESEARCHERID
THOMSON REUTERS
[Kalita, 6(6): June 2019] ISSN 2348 – 8034
DOI- 10.5281/zenodo.3246627 Impact Factor- 5.070

Result 3.13: Determinants and spectrums of Gram matrices of the 10 CATs obtained in theorem 3.12 are calculated and listed in table 3. This table also shows the spaces in which these 10 CATs tetrahedrons exist.

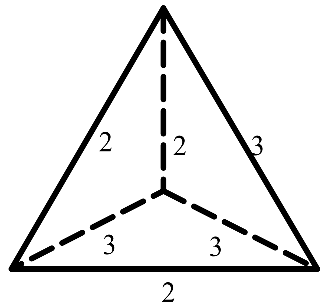
Table 3

CATs	Determinant	Spectrum	Space (Ref. Def. 3.1)
$T_{0n}\text{-}1=[2,2,2,2,2,2]$	1	$(1,1,1,1)$	Spherical
$T_{0n}\text{-}2=[2,2,2,3,2,2]$	$\frac{3}{4}$	$\left(\frac{3}{2},\frac{1}{2},1,1\right)$	Spherical
$T_{0n}\text{-}3=[2,2,2,4,2,2]$	$\frac{1}{2}$	$\left(1+\frac{1}{2}\sqrt{2},1-\frac{1}{2}\sqrt{2},1,1\right)$	Spherical
$T_{0n}\text{-}4=[2,2,2,5,2,2]$	$1-\cos^2\left(\frac{\pi}{5}\right)$	$\begin{pmatrix}1+\frac{1}{2}\sqrt{1+2\cos\left(\frac{\pi}{5}\right)},\\ 1-\frac{1}{2}\sqrt{1+2\cos\left(\frac{\pi}{5}\right)},\\ 1,\\ 1\end{pmatrix}$	Spherical
$T_{0n}\text{-}5=[2,2,2,3,2,3]$	$\frac{1}{2}$	$\left(1+\frac{1}{2}\sqrt{2},1-\frac{1}{2}\sqrt{2},1,1\right)$	Spherical
$T_{0n}\text{-}6=[2,2,2,4,2,3]$	$\frac{1}{4}$	$\left(1+\frac{1}{2}\sqrt{3},1-\frac{1}{2}\sqrt{3},1,1\right)$	Spherical
$T_{0n}\text{-}7=[2,2,2,5,2,3]$	$\frac{3}{4}-\cos^2\left(\frac{\pi}{5}\right)$	$\begin{pmatrix}1+\frac{1}{2}\sqrt{2+2\cos\left(\frac{\pi}{5}\right)},\\ 1-\frac{1}{2}\sqrt{2+2\cos\left(\frac{\pi}{5}\right)},\\ 1,\\ 1\end{pmatrix}$	Spherical
$T_{0n}\text{-}8=[2,2,2,3,3,3]$	$\frac{1}{4}$	$\left(1+\frac{1}{2}\sqrt{3},1-\frac{1}{2}\sqrt{3},1,1\right)$	Spherical
$T_{0n}\text{-}9=[2,2,2,4,3,3]$	0	$(0,2,1,1)$	Euclidean
$T_{0n}\text{-}10=[2,2,2,5,3,3]$	$\frac{1}{2}-\cos^2\left(\frac{\pi}{5}\right)$	$\begin{pmatrix}1+\frac{1}{2}\sqrt{3+2\cos\left(\frac{\pi}{5}\right)},\\ 1-\frac{1}{2}\sqrt{3+2\cos\left(\frac{\pi}{5}\right)},\\ 1,\\ 1\end{pmatrix}$	Hyperbolic

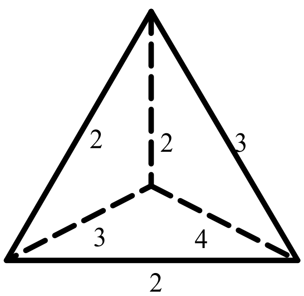
Theorem 3.14: [28] If a CAT T has no edge of order $n \geq 6$, then there are exactly 8 such T upto symmetry with at least one vertex is of order $(2,2,3)$ and no vertex is of order $(2,2,2)$.



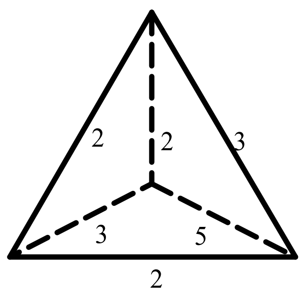
$T_{0n}\text{-}11 = [2,2,3,2,2,3]$



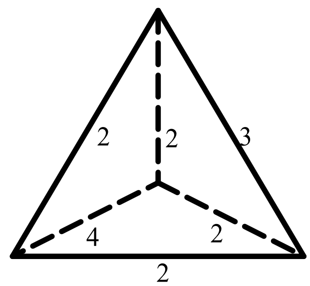
$T_{0n}\text{-}12 = [2,2,3,3,2,3]$



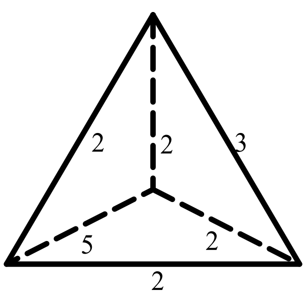
$T_{0n}\text{-}13 = [2,2,3,4,2,3]$



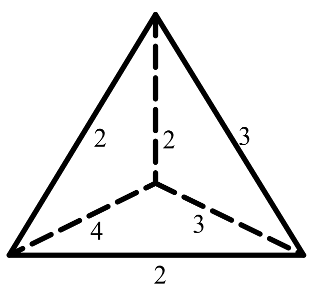
$T_{0n}\text{-}14 = [2,2,3,5,2,3]$



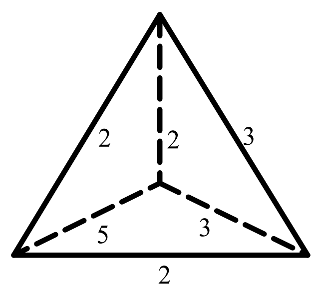
$T_{0n}\text{-}15 = [2,2,3,2,2,4]$



$T_{0n}\text{-}16 = [2,2,3,2,2,5]$



$T_{0n}\text{-}17 = [2,2,3,3,2,4]$



$T_{0n}\text{-}18 = [2,2,3,3,2,5]$

Figure 9

Result 3.15: Determinants and spectrums of gram matrices of the 8 CATs obtained in theorem 3.14 are calculated and listed in table 4. This table also shows the spaces in which these 8 CATs tetrahedrons exist.

Table 4

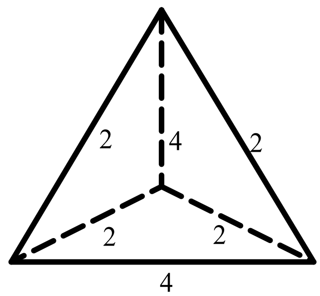
CATs	Determinant	Spectrum	Space(Ref. Def. 3.1)
$T_{0n}\text{-}11 = [2,2,3,2,2,3]$	$\frac{9}{16}$	$\left(\frac{3}{2},\frac{1}{2},\frac{3}{2},\frac{1}{2}\right)$	Spherical

ISSN 2348 – 8034
Impact Factor- 5.070

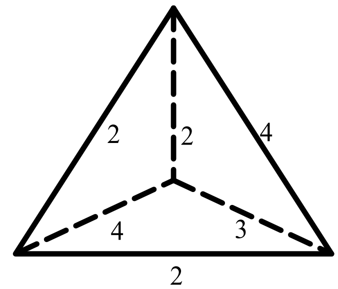
$T_{0n}\text{-}12=[2,2,3,3,2,3]$	$\frac{5}{16}$	$\left(\begin{array}{l}\frac{3}{4}+\frac{1}{4}\sqrt{5},\frac{3}{4}-\frac{1}{4}\sqrt{5},\\ \frac{5}{4}+\frac{1}{4}\sqrt{5},\frac{5}{4}-\frac{1}{4}\sqrt{5}\end{array}\right)$	Spherical
$T_{0n}\text{-}13=[2,2,3,4,2,3]$	$\frac{1}{16}$	$\left(\begin{array}{l}1-\frac{1}{4}\sqrt{6}-\frac{1}{4}\sqrt{2},1+\frac{1}{4}\sqrt{6}+\frac{1}{4}\sqrt{2},\\ 1-\frac{1}{4}\sqrt{6}+\frac{1}{4}\sqrt{2},1+\frac{1}{4}\sqrt{6}-\frac{1}{4}\sqrt{2}\end{array}\right)$	Spherical
$T_{0n}\text{-}14=[2,2,3,5,2,3]$	$\frac{9}{16}-\cos^2\left(\frac{\pi}{5}\right)$	$\left(\begin{array}{l}1+\frac{1}{2}\cos\left(\frac{\pi}{5}\right)+\frac{1}{2}\sqrt{1+\cos^2\left(\frac{\pi}{5}\right)},\\ 1+\frac{1}{2}\cos\left(\frac{\pi}{5}\right)-\frac{1}{2}\sqrt{1+\cos^2\left(\frac{\pi}{5}\right)},\\ 1-\frac{1}{2}\cos\left(\frac{\pi}{5}\right)+\frac{1}{2}\sqrt{1+\cos^2\left(\frac{\pi}{5}\right)},\\ 1-\frac{1}{2}\cos\left(\frac{\pi}{5}\right)+\frac{1}{2}\sqrt{1-\cos^2\left(\frac{\pi}{5}\right)}\end{array}\right)$	Hyperbolic
$T_{0n}\text{-}15=[2,2,3,2,2,4]$	$\frac{3}{8}$	$\left(\frac{1}{2},\frac{3}{2},1+\frac{1}{2}\sqrt{2},1-\frac{1}{2}\sqrt{2}\right)$	Spherical
$T_{0n}\text{-}16=[2,2,3,2,2,5]$	$\frac{3}{4}-\frac{3}{4}\cos^2\left(\frac{\pi}{5}\right)$	$\left(1+\cos\left(\frac{\pi}{5}\right),1-\cos\left(\frac{\pi}{5}\right),\frac{3}{2},\frac{1}{2}\right)$	Spherical
$T_{0n}\text{-}17=[2,2,3,3,2,4]$	$\frac{1}{8}$	$\left(\begin{array}{l}1-\frac{1}{2}\sqrt{2+\sqrt{2}},1+\frac{1}{2}\sqrt{2+\sqrt{2}},\\ 1-\frac{1}{2}\sqrt{2-\sqrt{2}},1+\frac{1}{2}\sqrt{2-\sqrt{2}}\end{array}\right)$	Spherical
$T_{0n}\text{-}18=[2,2,3,3,2,5]$	$\frac{1}{2}-\frac{3}{4}\cos^2\left(\frac{\pi}{5}\right)$	$\left(\begin{array}{l}1+\frac{1}{4}\sqrt{6+2\sqrt{4\cos^2\left(\frac{\pi}{5}\right)+4\cos\left(\frac{\pi}{5}\right)+5+4\cos\left(\frac{\pi}{5}\right)}},\\ 1-\frac{1}{4}\sqrt{6+2\sqrt{4\cos^2\left(\frac{\pi}{5}\right)+4\cos\left(\frac{\pi}{5}\right)+5+4\cos\left(\frac{\pi}{5}\right)}},\\ 1+\frac{1}{4}\sqrt{6-2\sqrt{4\cos^2\left(\frac{\pi}{5}\right)+4\cos\left(\frac{\pi}{5}\right)+5+4\cos\left(\frac{\pi}{5}\right)}},\\ 1-\frac{1}{4}\sqrt{6-2\sqrt{4\cos^2\left(\frac{\pi}{5}\right)+4\cos\left(\frac{\pi}{5}\right)+5+4\cos\left(\frac{\pi}{5}\right)}}\end{array}\right)$	Spherical

Theorem 3.16: [28] If a CAT T has no edge of order $n\geq 6$, then there are exactly 4 such T upto symmetry with at least one vertex is of order $(2,2,4)$ and no vertex is of order of the forms $(2,2,2),(2,2,3)$.

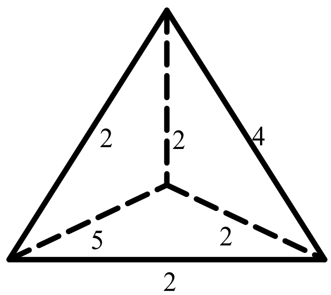
[Kalita, 6(6): June 2019] ISSN 2348 – 8034
DOI- 10.5281/zenodo.3246627 Impact Factor- 5.070



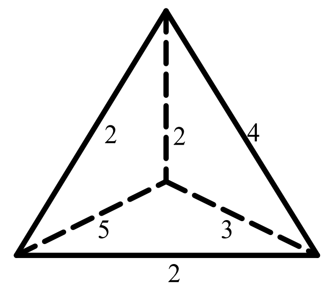
$T_{0n}\text{-}19=[2,4,2,2,4,2]$



$T_{0n}\text{-}20=[2,2,4,3,2,4]$



$T_{0n}\text{-}21=[2,2,4,2,2,5]$



$T_{0n}\text{-}22=[2,2,4,3,2,5]$

Figure 10

Result 3.17: Determinants and spectrums of gram matrices of the 4 CATs obtained in theorem 3.16 are calculated and listed in table 5. This table also shows the spaces in which these 4 CATs tetrahedrons exist.

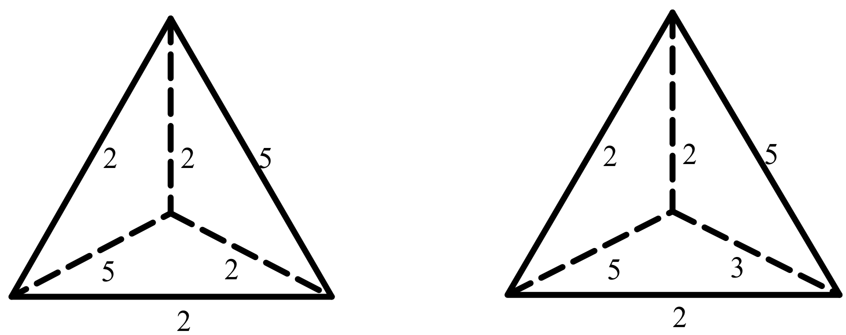
Table 5

CATs	Determinant	Spectrum	Space (Ref. Def. 3.1
$T_{0n}\text{-}19=[2,4,2,2,4,2]$	$\frac{1}{4}$	$\left(\begin{array}{l}1+\frac{1}{2}\sqrt{2},1-\frac{1}{2}\sqrt{2},\\1+\frac{1}{2}\sqrt{2},1-\frac{1}{2}\sqrt{2}\end{array}\right)$	Spherical
$T_{0n}\text{-}20=[2,2,4,3,2,4]$	0	$\left(0,2,\frac{1}{2},\frac{3}{2}\right)$	Euclidean
$T_{0n}\text{-}21=[2,2,4,2,2,5]$	$\frac{1}{2}-\frac{1}{2}\cos^2\left(\frac{\pi}{5}\right)$	$\left(\begin{array}{l}1+\frac{1}{2}\sqrt{2},1-\frac{1}{2}\sqrt{2},\\1+\cos\left(\frac{\pi}{5}\right),1-\cos\left(\frac{\pi}{5}\right)\end{array}\right)$	Spherical

ISSN 2348 – 8034
Impact Factor- 5.070

$T_{0n}\text{-}22=[2,2,4,3,2,5]$	$\frac{1}{4}-\frac{1}{2}\cos^2\left(\frac{\pi}{5}\right)$	$\begin{pmatrix} 1+\frac{1}{2}\sqrt{2+\sqrt{\cos^2\left(\frac{\pi}{5}\right)+2+\cos\left(\frac{\pi}{5}\right)}}, \\ 1-\frac{1}{2}\sqrt{2+\sqrt{\cos^2\left(\frac{\pi}{5}\right)+2+\cos\left(\frac{\pi}{5}\right)}}, \\ 1+\frac{1}{2}\sqrt{2-\sqrt{\cos^2\left(\frac{\pi}{5}\right)+2+\cos\left(\frac{\pi}{5}\right)}}, \\ 1-\frac{1}{2}\sqrt{2-\sqrt{\cos^2\left(\frac{\pi}{5}\right)+2+\cos\left(\frac{\pi}{5}\right)}} \end{pmatrix}$	Hyperbolic

Theorem 3.18: [28] If a CAT T has no edge of order $n \geq 6$, then there are exactly 2 such T upto symmetry with at least one vertex is of order $(2,2,5)$ and no vertex is of order of the forms $(2,2,2),(2,2,3),(2,2,4)$.



$T_{0n}\text{-}23=[2,2,5,2,2,5]$ $T_{0n}\text{-}24=[2,2,5,3,2,5]$

Figure 11

Result 3.19: Determinants and spectrums of gram matrices of the 2 CATs obtained in theorem 3.18 are calculated and listed in table 6. This table also shows the spaces in which these 2 CATs tetrahedrons exist.

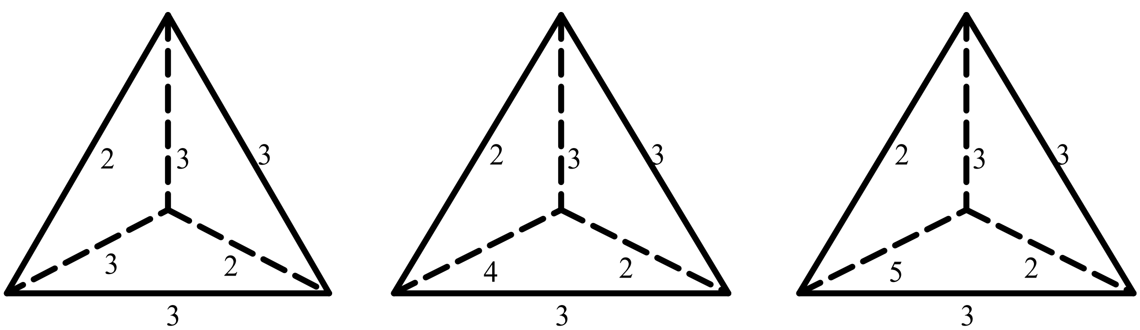
Table 6

CATs	Determinant	Spectrum	Space (Ref. Def. 3.1)
$T_{0n}\text{-}23=[2,2,5,2,2,5]$	$1-2\cos^2\left(\frac{\pi}{5}\right)+\cos^4\left(\frac{\pi}{5}\right)$	$\begin{pmatrix} 1+\frac{1}{2}\sqrt{1+2\cos\left(\frac{\pi}{5}\right)}, \\ 1-\frac{1}{2}\sqrt{1+2\cos\left(\frac{\pi}{5}\right)}, \\ 1+\frac{1}{2}\sqrt{1+2\cos\left(\frac{\pi}{5}\right)}, \\ 1-\frac{1}{2}\sqrt{1+2\cos\left(\frac{\pi}{5}\right)} \end{pmatrix}$	Spherical

ISSN 2348 – 8034
Impact Factor- 5.070

$T_{0n}\text{-}24=[2,2,5,3,2,5]$	$\frac{3}{4}-2\cos^2\left(\frac{\pi}{5}\right)+\cos^4\left(\frac{\pi}{5}\right)$	$\left(\frac{5}{4}+\frac{1}{4}\sqrt{5+8\cos\left(\frac{\pi}{5}\right)}, \frac{5}{4}-\frac{1}{4}\sqrt{5+8\cos\left(\frac{\pi}{5}\right)}, \frac{5}{4}+\frac{1}{4}\sqrt{5+8\cos\left(\frac{\pi}{5}\right)}, \frac{5}{4}-\frac{1}{4}\sqrt{5+8\cos\left(\frac{\pi}{5}\right)}\right)$	Hyperbolic

Theorem 3.20: [28] If a CAT T has no edge of order $n \geq 6$, then there are exactly 3 such T upto symmetry with at least one vertex is of order $(2,3,3)$ and no vertex is of order of the forms $(2,2,2),(2,2,3),(2,2,4),(2,2,5)$.



$T_{0n}\text{-}25=[2,3,3,2,3,3]$ $T_{0n}\text{-}26=[2,3,3,2,3,4]$ $T_{0n}\text{-}27=[2,3,3,2,3,5]$

Figure 12

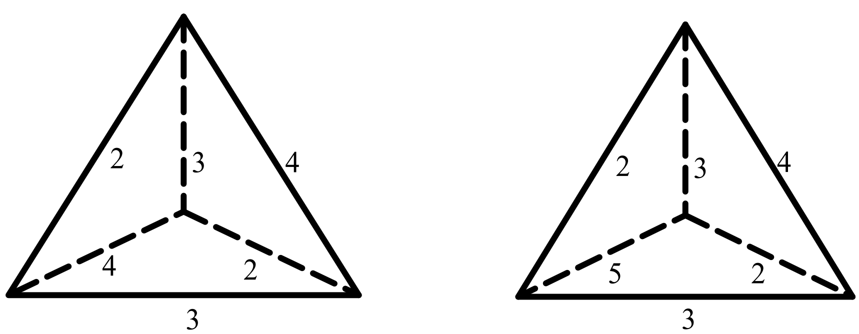
Result 3.21: Determinants and spectrums of gram matrices of the 3 CATs obtained in theorem 3.20 are calculated and listed in table 7. This table also shows the spaces in which these 3 CATs tetrahedrons exist.

Table 7

CAT	Determinant	Spectrum	Space (Ref. Def. 3.1)
$T_{0n}\text{-}25=[2,3,3,2,3,3]$	0	$(0,2,1,1)$	Euclidean
$T_{0n}\text{-}26=[2,3,3,2,3,4]$	$-\frac{1}{8}\left(\frac{1}{2}+\sqrt{2}\right)$	$\left(\frac{5}{4}+\frac{1}{4}\sqrt{2}+\frac{1}{4}\sqrt{7-2\sqrt{2}}, \frac{5}{4}+\frac{1}{4}\sqrt{2}-\frac{1}{4}\sqrt{7-2\sqrt{2}}, \frac{3}{4}-\frac{1}{4}\sqrt{2}+\frac{1}{4}\sqrt{7-2\sqrt{2}}, \frac{3}{4}-\frac{1}{4}\sqrt{2}-\frac{1}{4}\sqrt{7-2\sqrt{2}}\right)$	Hyperbolic

$\mathrm{T_{0n}}\text{-}27=[2,3,3,2,3,5]$	$\frac{5}{16}-\frac{1}{4}\cos\left(\frac{\pi}{5}\right)-\frac{3}{4}\cos^2\left(\frac{\pi}{5}\right)$	$\left(\begin{array}{l}\frac{5}{4}+\frac{1}{2}\cos\left(\frac{\pi}{5}\right)+\frac{1}{4}\sqrt{5-4\cos\left(\frac{\pi}{5}\right)+4\cos^2\left(\frac{\pi}{5}\right)},\\ \frac{5}{4}+\frac{1}{2}\cos\left(\frac{\pi}{5}\right)-\frac{1}{4}\sqrt{5-4\cos\left(\frac{\pi}{5}\right)+4\cos^2\left(\frac{\pi}{5}\right)},\\ \frac{3}{4}-\frac{1}{2}\cos\left(\frac{\pi}{5}\right)+\frac{1}{4}\sqrt{5-4\cos\left(\frac{\pi}{5}\right)+4\cos^2\left(\frac{\pi}{5}\right)},\\ \frac{3}{4}-\frac{1}{2}\cos\left(\frac{\pi}{5}\right)-\frac{1}{4}\sqrt{5-4\cos\left(\frac{\pi}{5}\right)+4\cos^2\left(\frac{\pi}{5}\right)}\end{array}\right)$	Hyperbolic

Theorem 3.22: [28] If a CAT T has no edge of order $n\geq 6$, then there are exactly 2 such T upto symmetry with at least one vertex is of order $(2,3,4)$ and no vertex is of order of the forms $(2,2,2),(2,2,3),(2,2,4),(2,2,5),(2,3,3)$.



$\mathrm{T_{0n}}\text{-}28=[2,3,4,2,3,4]$ $\mathrm{T_{0n}}\text{-}29=[2,3,4,2,3,5]$

Figure 13

Result 3.23: Determinants and spectrums of gram matrices of the 2 CATs obtained in theorem 3.22 are calculated and listed in table 8. This table also shows the spaces in which these 2 CATs tetrahedrons exist.

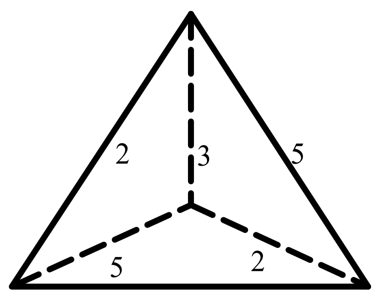
Table 8

CATs	Determinant	Spectrum	Space (Ref. Def. 3.1)
$\mathrm{T_{0n}}\text{-}28=[2,3,4,2,3,4]$	$-\frac{7}{6}$	$\left(\begin{array}{l}\frac{1}{2}+\frac{1}{2}\sqrt{2},\frac{1}{2}-\frac{1}{2}\sqrt{2},\\ \frac{3}{2}+\frac{1}{2}\sqrt{2},\frac{3}{2}-\frac{1}{2}\sqrt{2}\end{array}\right)$	Hyperbolic

 ISSN 2348 – 8034
Impact Factor- 5.070

$T_{0n}\text{-}29=\left[2,3,4,2,3,5\right]$	$\frac{1}{16}-\frac{1}{4}\sqrt{2}\cos\left(\frac{\pi}{5}\right)-\frac{1}{2}\cos^{2}\left(\frac{\pi}{5}\right)$	$\left(\begin{array}{l}\frac{9}{8}+\frac{1}{4}\sqrt{2}+\frac{1}{8}\sqrt{5}+\frac{1}{8}\sqrt{30-4\sqrt{2}+2\sqrt{5}-4\sqrt{2}\sqrt{5}},\\ \frac{9}{8}+\frac{1}{4}\sqrt{2}+\frac{1}{8}\sqrt{5}-\frac{1}{8}\sqrt{30-4\sqrt{2}+2\sqrt{5}-4\sqrt{2}\sqrt{5}},\\ \frac{7}{8}-\frac{1}{4}\sqrt{2}-\frac{1}{8}\sqrt{5}+\frac{1}{8}\sqrt{30-4\sqrt{2}+2\sqrt{5}-4\sqrt{2}\sqrt{5}},\\ \frac{7}{8}-\frac{1}{4}\sqrt{2}-\frac{1}{8}\sqrt{5}-\frac{1}{8}\sqrt{30-4\sqrt{2}+2\sqrt{5}-4\sqrt{2}\sqrt{5}}\end{array}\right)$	Hyperbolic

Theorem 3.24: [28] If a CAT T has no edge of order $n\geq 6$, then there are exactly 1 such T upto symmetry with at least one vertex is of order $(2,3,5)$ and no vertex is of order of the forms $(2,2,2),(2,2,3),(2,2,4),(2,2,5)$, $(2,3,3),(2,3,4)$.



$T_{0n}\text{-}30=\left[2,3,5,2,3,5\right]$

Figure 14

Result 3.25: Determinants and spectrums of gram matrices of the 1 CAT obtained in theorem 3.24 is calculated and listed in table 9. This table also shows the spaces in which these 1 CAT exists.

Table 9

CAT	Determinant	Spectrum	Space (Ref. Def. 3.1)
$T_{0n}\text{-}30=\left[2,3,5,2,3,5\right]$	$\frac{9}{16}-\frac{5}{2}\cos^{2}\left(\frac{\pi}{5}\right)+\cos^{4}\left(\frac{\pi}{5}\right)$	$\left(\begin{array}{l}\frac{3}{2}+\cos\left(\frac{\pi}{5}\right),\frac{1}{2}-\cos\left(\frac{\pi}{5}\right),\\ \frac{3}{2}-\cos\left(\frac{\pi}{5}\right),\frac{1}{2}+\cos\left(\frac{\pi}{5}\right)\end{array}\right)$	Hyperbolic

Remark 3.26: Out of 35 CATs, there are exactly 3 Euclidean CATs, each of them has no edge of order $n\geq 6$, $n\in$ upto symmetry. [Refer tables 3, 5 and 7]

Remark 3.27: Out of 35 CATs, there are exactly 23 spherical CATs, 1 CAT has two edges of order $n\geq 6$, 4 CATs have one edge of order $n\geq 6$ and rest 18 CATs have no edge of order $n\geq 6$, $n\in$ upto symmetry. [Refer tables 1 to 6]

Remark 3.28: Out of 35 CATs, there are exactly 9 hyperbolic CATs, each of them has no edge of order $n \geq 6$, $n \in$ upto symmetry. [Refer tables 3 to 9]

Remark 3.29: Out of 35 CATs, there are exactly 9 hyperbolic CATs, and out of these 9, the last 3 hyperbolic CATs:

$$\mathrm{T_{0n}}\text{-}28 = [2,3,4,2,3,4], \mathrm{T_{0n}}\text{-}29 = [2,3,4,2,3,5], \mathrm{T_{0n}}\text{-}30 = [2,3,5,2,3,5]$$

are nothing but the 3 CHC (Compact Hyperbolic Coxeter) tetrahedrons found in article [29] upto symmetry. [Refer tables 8 and 9]

REFERENCES

1. *John G. Ratcliffe, Foundations of Hyperbolic Manifolds, ©1994 by Springer-Verlag, New York, Inc.*
2. *http://en.wikipedia.org/wiki/Triangle-group.*
3. *P. Kalita and B. Kalita, Properties of Coxeter Andreev's Tetrahedrons, IOSR Journal of Mathematics (IOSR-JM) e-ISSN: 2278-3008, p-ISSN: 2319-7676. Volume 9, Issue 6, 2014, PP 81-105.*
4. *P. Kalita, "Gram Spectrums of Triangles with Triangle Groups", IJMA-5(6), 2014, 64-71.*
5. *Dipankar Mondal, Introduction to Reflection Groups, Triangle Group (Course Project), April 26, 2013.*
6. *H. S. M. Coxeter, Discrete groups generated by reflections, Ann. Of Math. 35 (1934), 588-621.*
7. *Wilhelm Magnus, Noneuclidean Tesselations and Their Groups. Academic Press New York and London, 1974.*
8. *Alexandre V. Borovik, Anna Borovik, Mirror and Reflection: The Geometry of Finite Reflection Groups. Springer-Universitext, 2010.*
9. *en.wikipedia.org/wiki/simplex.*
10. *J. Mcleod, Hyperbolic Coxeter Pyramids, Advances in Pure Mathematics, Scientific Research, 2013, 3, 78- 82*
11. *Roland K. W. Roeder, Compact hyperbolic tetrahedra with non-obtuse dihedral angles, August 10, 2013, arxiv.org/pdf/math/0601148.*
12. *Aleksandr Kolpakov, On extremal properties of hyperbolic coxeter polytopes and their reflection groups, Thesis No: 1766, e-publi.de, 2012.*
13. *Anna Felikson, Pavel Tumaarkin, Coxeter polytopes with a unique pair of non intersecting facets, Journal of Combinatorial Theory, Series A 116 (2009) 875-902.*
14. *Pavel Tumarkin, Compact Hyperbolic Coxeter polytopes with facets, The Electronic Journal of Combinatorics 14 (2007).*
15. *Roland K.W. Roeder, John H. Hubbard and William D. Dunbar, Andreev's Theorem on Hyperbolic Polyhedra, Ann. Inst. Fourier, Grenoble 57, 3 (2007), 825-882.*
16. *D. Cooper, D. Long and M. Thistlethwaite, Computing varieties of representations ofhyperbolic 3- manifolds into , Experiment. Math. 15, 291-305 (2006)*
17. *D. A. Derevnin, A. D. Mednykh and M. G. Pashkevich, On the volume of symmetric tetrahedron, Siberian Mathematical Journal, Vol. 45, No. 5, pp. 840-848, 2004*
18. *Yunhi Cho and Hyuk Kim. On the volume formula for hyperbolic tetrahedral. Discrete Comput. Geom., 22(3): 347-366, 1999.*
19. *E. B. Vinberg, Hyperbolic Reflection Groups, Uspekhi Mat. Nauk 40, 29-66 (1985)*
20. *R. Guo and Y. Wang, Eigenvalues of Gram Matrices.*
21. *E. B. Vinberg, The absence of crystallographic groups of reflections in Lobachevskij spaces of large dimensions, Trans. Moscow Math. Soc. 47 (1985), 75-112.*
22. *M. Hazewinkel, at .el, "The Ubiquity of Coxeter-Dynkin Diagrams", Nieuw Archief Voor Wiskunde (3), XXV(1977), 257-307.*
23. *en.wikipedia.org/wiki/tetrahedron#Applications access in October, 2013.*
24. *J. Mcleod, Hyperbolic Coxeter Pyramids, Advances in Pure Mathematics, Scientific Research, 2013, 3, 78-82*
25. *Roland K. W. Roeder, Compact hyperbolic tetrahedra with non-obtuse dihedral angles, August 10, 2013, arxiv.org/pdf/math/0601148.*
26. *D. A. Derevnin, A. D. Mednykh and M. G. Pashkevich, On the volume of symmetric tetrahedron, Siberian Mathematical Journal, Vol. 45, No. 5, pp. 840-848, 2004*
27. *E. B. Vinberg, The absence of crystallographic groups of reflections in Lobachevskij spaces of large dimensions, Trans. Moscow Math. Soc. 47 (1985), 75-112.*

28. *P. Kalita and B. Kalita, Properties of Coxeter Andreev's Tetrahedrons, ISOR Journal of Mathematics, 2014, PP 81-105.*
29. *P. Kalita and B. Kalita, Complete Set of CHC Tetrahedrons, International Journal of Computer Applications, Vol. 87, No. 11, 2014.*